

Causality Preserving Information Transfer Measure for Control Dynamical System

Subhrajit Sinha and Umesh Vaidya

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Abstract

In this paper, we show through examples, how the existing definitions of information transfer, namely directed information and transfer entropy fail to capture true causal interaction between states in control dynamical system. We propose a new definition of information transfer, based on the ideas from dynamical system theory, and show that this new definition can capture true causal interaction between states. The information transfer measure is generalized to define transfers between the various signals in a control dynamical system and analytical expression for information transfer between state-to-state, input-to-state, state-to-output, and input-to-output are provided for linear systems. There is a natural extension of our proposed definition to define information transfer over n time steps and average information transfer over infinite time step. We show that the average information transfer in feedback control system between plant output and input is equal to the entropy of the open loop dynamics thereby re-deriving the Bode fundamental limitation results using the proposed definition of transfer.

1 Introduction

From the days of Aristotle, philosophers and scientists have been concerned with the notion of causality. As far back as 300 B.C. Aristotle had realized the importance of causality and even after about two and half thousand years, there is no universally accepted definition of causality. But causality and influence characterization is a subject of immense importance and finds its application in many different applications like world wide web and social media, biological networks, neural science, economics, finance etc. Usually, concepts of information theory are used in such applications and a study of the information flow

*Financial support from the National Science Foundation grant ECCS-1150405 and CNS-1329915 is gratefully acknowledged. S. Sinha and U. Vaidya is with the Department of Electrical & Computer Engineering, Iowa State University, Ames, IA 50011 ugvaidya@iastate.edu

between the components of the network throws light on causality and the influential nodes of the network. In [1] the authors use information based metric to characterize the most influential nodes in social networks. In neuroscience, concepts of information theory are used to understand how information flows in different parts of the brain [2] and identifying influence in gene regulatory networks [3, 4]. In economic and financial networks, information transfer can be used to infer causal interactions from the time series data [5, 6, 7]. In control theoretic setting, in [8], the authors have used information theoretic ideas to study the classical concepts of control like controllability and observability.

Causality characterization was initially geared towards time series data and Granger causality [5],[7], directed information [9],[10] and Schreiber's transfer entropy [11] have been the most popular tools used for inferring the causality structure and influence characterization. In dynamical system setting, it was Liang and Kleeman [12],[13] who introduced the concept of information transfer between the states and used it for predictability analysis. The formulation of information transfer used in this paper is in dynamical systems setting and is closely related to and inspired from information transfer framework developed in [14, 12, 15] for nonlinear dynamical systems. We used the ideas of Liang-Kleeman transfer to propose an axiomatic definition of information transfer in discrete linear dynamic network [16]. The axioms were physically motivated and it was shown in [16] that there exists a unique expression for information transfer in a dynamic network satisfying these axioms. In [16], we had used absolute entropy to characterize the information flow. However, motivated by the definitions of directed information and transfer entropy, in this work we use conditional entropy instead of absolute entropy to characterize the information flow. Motivation for this work lies in the fact that, in dynamical systems, directed information fails to capture the intuitions of information transfer, and thus provide erroneous conclusion about the causal structure in a dynamical system. We show that our definition of information transfer does capture the correct causal structure of the system. We show, how this definition of information transfer between the states of a dynamical system can be easily extended to study the the information transfer between the inputs and outputs in a control dynamical system. In fact, this natural extension leads us to connect the information transfer to the Bode integral of the sensitivity transfer function from the output to the input in a feedback control system.

The paper is organised as follows. In section 2, we discuss, through some examples, how directed information fails to capture the notions of causality (zero transfer and indirect influence). In section 3 we provide our definition of information transfer. We also define n -step information transfer and average information transfer and for linear systems, we provide explicit formulas for computing the transfer. We also revisit the examples in section 2 and show how our definition of information transfer captures the intuitions of information transfer. In section 4, we generalize our definition of information transfer to define information transfer between the inputs and outputs in a linear control dynamical system. We also study the information transfer in a feedback control system and show how this is related to the Bode integral of the sensitivity transfer function from the output to the input. This is followed by conclusions in section 5.

2 Directed Information as a Measure of Causality

Directed information and transfer entropy are two of the most popular notions of information transfer used to characterize causality. In this section, we show that these notions of information transfer cannot faithfully capture the true causality structure in control dynamical system.

Directed information was first defined by Massey [9] as a generalization of Marko's bidirectional information [17]. Both bidirectional information and directed information gave a sense of direction to Shannon's information theory and is viewed as a generalized information theory. Let $X^n = \{X_1, X_2, \dots, X_n\}$ and $Y^n = \{Y_1, Y_2, \dots, Y_n\}$ be two stochastic processes, viewed as a sequence of random variables. Massey and Kramer [10] defined the directed information from X^n to Y^n as

$$I(X^n \rightarrow Y^n) = H(Y^n) - H(Y^n \parallel X^n) \quad (1)$$

where $H(Y^n)$ is the entropy of the sequence Y^n and $H(Y^n \parallel X^n) := \sum_{i=1}^n H(Y_i | Y^{i-1}, X^i)$ is the entropy of Y^n *causally* conditioned on X^n .

The directed information is asymmetric and gives a directional sense to the information and defines a measure to determine the direction of information flow. Since the development of the concept of directed information, this has been used in many different applications, like determining the channel capacity of a communication channel. Moreover, for Gaussian variables, directed information and Granger causality [7], [5] are equivalent [18]. This allows the concept of directed information to be used as a measure of causality and hence it has been used to infer about the causal structure of statistical processes. In the following, we demonstrate using three different examples how the directed information fails to capture the true causality structure in dynamical systems setting. While the arguments are made in the context of directed information, similar conclusions can be drawn in the context of transfer entropy. We claim that at the heart of the problem is the manner in which causal conditioning is performed in both these definitions of information transfer.

2.1 Examples

Example 1. Consider the following linear system with output

$$\begin{aligned} \begin{pmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{pmatrix} &= \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \sigma \xi(k) \\ y(t) &= (0 \ 1 \ 0) x(t) \end{aligned} \quad (2)$$

where $x_i(t)$ and $y(t)$ are the states and output respectively at time step t and $\xi(t)$ is a independent identically distributed (i.i.d.) zero mean unit variance Gaussian noise and σ is a constant. We notice that while there is directed path from $x_2 \rightarrow x_1$ and $x_3 \rightarrow x_1$ (fig. 1(a)), there is no path from x_1 to x_2 or from x_1 to x_3 . Hence, we conclude that x_1 is not a cause of x_2 or x_3 , that is, x_1 is not influencing x_2 or x_3 . So if we treat the dynamical

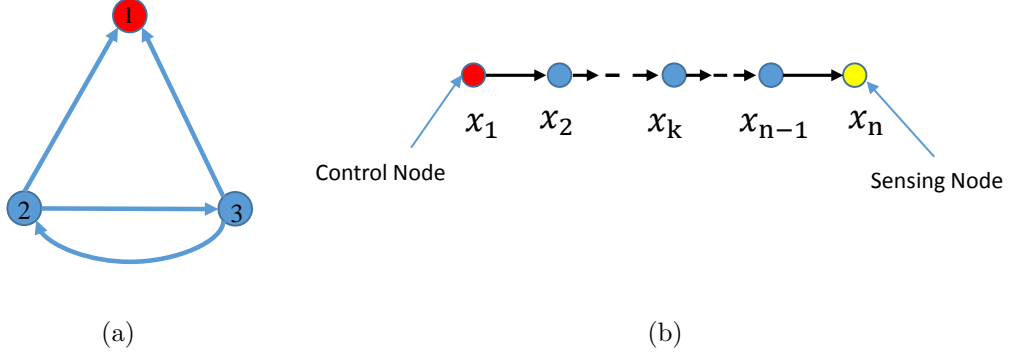


Figure 1: (a) Graph of dynamical system of example 1. (b) Graph of dynamical system of example 2.

system as a stochastic process, we expect that the flow of information from $x_1 \rightarrow x_2$ and $x_1 \rightarrow x_3$ to be zero, thereby inferring that there is no causal connection from $x_1 \rightarrow x_2$ and $x_1 \rightarrow x_3$. In fact closer examination reveals that x_1 is not influencing x_2 and x_3 over any number of time steps. However, as we show below, the directed information fails to capture this true causal interaction between state x_1, x_2 , and x_3 . In particular, we show that $I(x_1^n \rightarrow x_2^n) \neq 0$ and $I(x_1^n \rightarrow x_3^n) \neq 0$ for any n . In Fig. 2(a), we plot the directed information from $x_1 \rightarrow x_2$ and $x_1 \rightarrow x_3$ over different time steps.

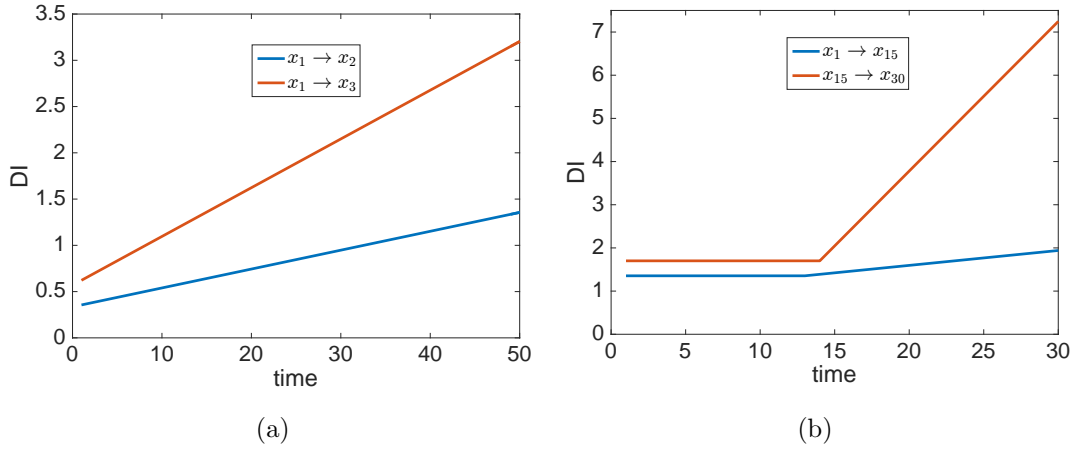


Figure 2: (a) Directed information plots for example 1. (b) Directed information plots for example 2.

Example 2. Next we consider a single input single output linear system

$$\begin{aligned} x(t+1) &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \\ y(t) &= (0 \quad \dots \quad 1) x(t) \end{aligned} \quad (3)$$

where $x(t) \in \mathbb{R}^N$. We note that the state x_1 is directly controlled through u and measurements are made at state x_N . State x_k , for $1 \ll k \ll N$, is assumed to be some intermittent state. Since the control input u does not affect state x_k instantaneously, but through series of k delays, we expect the information flow from state x_1 to x_k to be zero for time step $t = 0, \dots, k-1$. On the other hand since x_k influence x_N again through series of delay we expect the information flow from x_k to x_N to be zero for $t = k, \dots, N-k-1$. However as we show in Fig. 2(b), directed information from $x_1 \rightarrow x_k$ and $x_k \rightarrow x_N$ is nonzero for all time $t = 0, \dots$, conveying that there is instantaneous flow of information from x_1 to x_k and from x_k to x_N . The simulation results in Fig. 2(b) are obtained for $N = 30$ nodes with $k = 15$. This example demonstrates that the directed information fails to capture indirect path of influence.

The above examples reveal some serious deficiencies of directed information as a measure of causality in dynamical system setting. This suggest that the directed information is a measure of *statistical* interaction between two signals but it does not successfully capture the true *dynamical* interactions between dynamical states. We expect that any meaningful definition of information transfer in network dynamical system be able to capture true causal interaction and also obey time constraint of information flow. Our proposed definition of information transfer discussed in the following section precisely does that.

3 A New Definition of Information Transfer

Consider the following discrete time dynamical system,

$$z(t+1) = f(z(t)) + \xi(t) \quad (4)$$

where $z(t) \in \mathbb{R}^N$, $\xi(t) \in \mathbb{R}^N$ is assumed to vector valued random variable and $\xi(0), \xi(1), \dots$ are independent random vectors each having the same density g . The mapping $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is assumed to be at least continuous. Let $z = (z_1, \dots, z_N)^\top \in \mathbb{R}^N$. We are interested in defining the information transfer from state z_i to state z_j , as the system evolves from time step t to time step $t+1$. We denote this transfer by the notation $[T_{z_i \rightarrow z_j}]_t^{t+1}$.

We provide the following definition for information transfer from state z_i to state z_j for the dynamical system (4) :

Definition 3 (Information transfer). *The information transfer from z_i to z_j for dynamical system (4) as the system evolves from time t to time $t + 1$ and denoted by $[T_{z_i \rightarrow z_j}]_t^{t+1}$ is given by following formula*

$$[T_{z_i \rightarrow z_j}]_t^{t+1} = H(\rho(z_j(t+1)|z_j(t))) - H(\rho_{\#i}(z_j(t+1)|z_j(t))) \quad (5)$$

where $H(\rho(z_j)) = -\int_{\mathbb{R}^{|z_j|}} \rho(z_j) \log \rho(z_j) dz_j$ is the entropy of probability density function $\rho(z_j)$ and $H(\rho_{\#i}(z_j(t+1)|z_j(t)))$ is the entropy of $z_j(t+1)$, conditioned on $z_j(t)$, where z_i has been frozen.

The above definition of information transfer can be understood by rewriting the expression of information transfer as follows:

$$H(\rho(z_j(t+1)|z_j(t))) = [T_{z_i \rightarrow z_j}]_t^{t+1} + H(\rho_{\#i}(z_j(t+1)|z_j(t))) \quad (6)$$

so that the total entropy of z_j is the sum of transfer from z_i and the entropy of z_j , when z_i is absent.

3.1 n-step Information Transfer

The information transfer defined in (5) gives the information transferred from z_i to z_j as the dynamical system evolves from time step t to time step $t + 1$. So one can look at this definition as a *one-step* transfer. To generalize the one step transfer to *n-step* transfer, we define the *n-step* transfer from z_i to z_j as the system evolves from time step t to $t + n$, where $n \in \mathbb{Z}_{>0}$, as follows :

Definition 4 (*n-step* Information transfer). *The information transferred over n time steps, $n \in \mathbb{Z}_{>0}$, from z_i to z_j , which is denoted as $(T_{z_i \rightarrow z_j})_t^{t+n}$, as the dynamical system (4) evolves from time step t to $t + n$, is defined as*

$$(T_{z_i \rightarrow z_j})_t^{t+n} = H(\rho(z_j(t+n)|z_j(t+n-1) \cdots z_j(t))) - H(\rho_{\#i}(z_j(t+n)|z_j(t+n-1) \cdots z_j(t))) \quad (7)$$

where $H(\rho(z_j(t+n)|z_j(t+n-1) \cdots z_j(t)))$ is the conditional entropy of $z_j(t+n)$, conditioned on past n -steps of z_j and $H(\rho_{\#i}(z_j(t+n)|z_j(t+n-1) \cdots z_j(t)))$ is the conditional entropy of $z_j(t+n)$ on its past n -steps and the state z_i has been frozen (held constant) from time step t to time step $t + n$.

In general, we have the following

Theorem 5.

$$\sum_{i=1}^n (T_{z_i \rightarrow z_j})_t^{t+i} = H(z_j(t+n) \cdots z_j(t)) - H_{\#i}(z_j(t+n) \cdots z_j(t)) \quad (8)$$

where

$$H(z_j(t+n) \cdots z_j(t)) = H(\rho(z_j(t+n) \cdots z_j(t)))$$

and

$$H_{\#i}(z_j(t+n) \cdots z_j(t)) = H(\rho_{\#i}(z_j(t+n) \cdots z_j(t))).$$

Proof. We prove this by induction. For $i = 1$, using the fact that $H(y) = H_{\#}(y)$, we have $(T_{x \rightarrow y})_t^{t+1} = H(y', y) - H_{\#}(y', y)$.

Let this be true for $i = k$. Hence, we have,

$$\sum_{i=1}^k (T_{x \rightarrow y})_t^{t+i} = H(y(t+k) \cdots y(t)) - H_{\#}(y(t+k) \cdots y(t))$$

Now,

$$\begin{aligned} (T_{x \rightarrow y})_t^{t+k+1} &= H(y(t+k+1)|y(t+k) \cdots y(t)) - H_{\#}(y(t+k+1)|y(t+k) \cdots y(t)) \\ &= [H(y(t+k+1) \cdots y(t)) - H_{\#}(y(t+k+1) \cdots y(t))] \\ &\quad - [H(y(t+k) \cdots y(t)) - H_{\#}(y(t+k) \cdots y(t))] \\ &= [H(y(t+k+1) \cdots y(t)) - H_{\#}(y(t+k+1) \cdots y(t))] - \sum_{i=1}^k (T_{x \rightarrow y})_t^{t+i} \end{aligned}$$

Hence,

$$\sum_{i=1}^{k+1} (T_{x \rightarrow y})_t^{t+i} = H(y(t+k+1) \cdots y(t)) - H_{\#}(y(t+k+1) \cdots y(t))$$

Hence,

$$\sum_{i=1}^n (T_{x \rightarrow y})_t^{t+i} = H(y(t+n) \cdots y(t)) - H_{\#}(y(t+n) \cdots y(t))$$

□

After this point, for notational convenience, we will use the notation $\rho(y_t^{t+n})$ to denote the density of the joint distribution of $y(t+n)y(t+n-1) \cdots y(t)$, that is, $\rho(y_t^{t+n}) = \rho(y(t+n)y(t+n-1) \cdots y(t))$. When $t = 0$, we will simply use $\rho(y^n)$ to denote $\rho(y(n)y(n-1) \cdots y(0))$.

Definition 6 (Average Information Transfer). *The average information transfer from X to Y is defined as*

$$\bar{T}_{x \rightarrow y} = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=0}^T (T_{x \rightarrow y})_0^i \quad (9)$$

With this,

$$\bar{T}_{x \rightarrow y} = \lim_{n \rightarrow \infty} \frac{1}{n} [H(y^n) - H_{\#}(y^n)] \quad (10)$$

Hence, the average information transfer from X to Y is a difference of two entropy rates, namely, entropy rate of Y ($\lim_{n \rightarrow \infty} \frac{1}{n} H(y^n)$) and entropy rate of Y when X is frozen ($\lim_{n \rightarrow \infty} \frac{1}{n} H_{\#}(y^n)$).

The average information transfer and average directed information, however, coincide as time goes to infinity. Intuitively, this can be explained as follows. Both average information transfer and average directed information can be written as a difference of two terms, where the first term is the same for both of them. The difference lies in the fact that, our definition of information transfer uses freezing, whereas, in directed information, one uses conditioning. However, as time goes to infinity, infinite step freezing and infinite step conditioning is the same and hence they converge to the same value. For example,

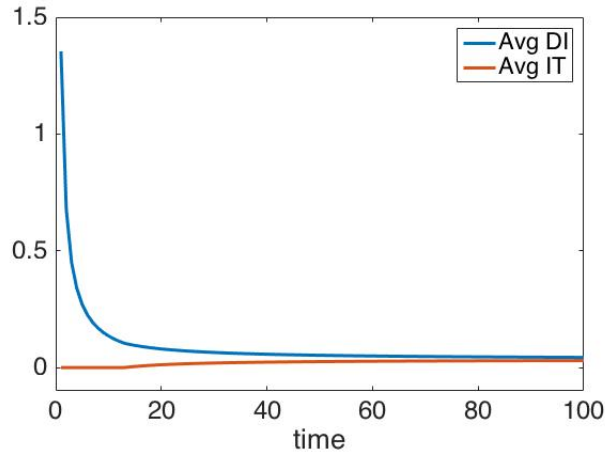


Figure 3: Comparison of average information transfer and average directed information for example (2).

lets consider the example (2). In figure 3, we plot the average information transfer and average directed information from x_1 to x_{30} . From the figure, we see that though average information transfer and average directed information differs in the initial stages, as the system evolves, they both converge to the same value.

3.2 Information transfer in linear dynamical system

The Eq. (5) provides formula for information transfer in general nonlinear system however for a special class of linear system analytical expression for the information transfer can be obtained. Consider the following linear time invariant dynamical system

$$z(t+1) = Az(t) + \sigma\xi(t) \quad (11)$$

where $z(t) \in \mathbb{R}^N$ and $\xi(t)$ is vector valued Gaussian random variable with zero mean and unit variance. We assume that the initial conditions are Gaussian with covariance $\Sigma(0)$. Since the system is linear, the distribution of the system state for all future time will remain Gaussian with covariance $\Sigma(t)$ satisfying

$$A\Sigma(t-1)A^\top + \sigma^2 I = \Sigma(t)$$

To define the information transfer between various subspace we introduce following notation to split the A matrix :

$$z(t+1) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} A_x & A_{xy} \\ A_{yx} & A_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \sigma \xi \quad (12)$$

The A matrix can be further split using the subspace decomposition $x = (x_1^\top, x_2^\top)^\top$ as follows:

$$\begin{pmatrix} A_x & A_{xy} \\ A_{yx} & A_y \end{pmatrix} = \begin{pmatrix} A_{x_1} & A_{x_1 x_2} & A_{x_1 y} \\ A_{x_2 x_1} & A_{x_2} & A_{x_2 y} \\ A_{yx_1} & A_{yx_2} & A_y \end{pmatrix} \quad (13)$$

Based on the decomposition of the system A matrix we can also decompose the covariance matrix Σ at time instant t as follows.

$$\Sigma = \begin{pmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{xy}^\top & \Sigma_y \end{pmatrix} = \begin{pmatrix} \Sigma_{x_1} & \Sigma_{x_1 x_2} & \Sigma_{x_1 y} \\ \Sigma_{x_1 x_2}^\top & \Sigma_{x_2} & \Sigma_{x_2 y} \\ \Sigma_{x_1 y}^\top & \Sigma_{x_2 y}^\top & \Sigma_y \end{pmatrix} \quad (14)$$

Using the above notation, we state following theorem providing explicit expression for information transfer in linear dynamical system during transient and steady state.

Theorem 7. *Consider the linear dynamical system (11) and associated splitting of state space in Eqs. (12) and (13). We have following expression for information transfer between various subspace*

$$[T_{x \rightarrow y}]_t^{t+1} = \frac{1}{2} \log \frac{|A_{yx} \Sigma_y^s(t) A_{yx}^\top + \sigma^2 I|}{\sigma^2} \quad (15)$$

where $\Sigma_y^s(t) = \Sigma_x(t) - \Sigma_{xy}(t) \Sigma_y(t)^{-1} \Sigma_{xy}(t)^\top$ is the Schur complement of $\Sigma_y(t)$ in the matrix $\Sigma(t)$.

$$[T_{x_1 \rightarrow y}]_t^{t+1} = \frac{1}{2} \log \frac{|A_{yx} \Sigma_y^s(t) A_{yx}^\top + \sigma^2 I|}{|A_{yx_2} (\Sigma_y^s)_{yx_2}(t) A_{yx_2}^\top + \sigma^2 I|} \quad (16)$$

where $|\cdot|$ is the determinant and $(\Sigma_y^s)_{yx_2}$ is the Schur complement of Σ_y in the matrix

$$\begin{pmatrix} \Sigma_{x_2} & \Sigma_{x_2 y} \\ \Sigma_{x_2 y}^\top & \Sigma_y \end{pmatrix}$$

Proof. From the formula of information transfer we have

$$[T_{x \rightarrow y}]_t^{t+1} = H(\rho(y'|y)) - H_{\mathcal{F}}(\rho(y'|y))$$

So to compute the transfer, we need to know $\rho(y'|y) = \frac{\rho(y',y)}{\rho(y)}$ and $\rho_{\mathcal{F}}(y'|y)$. Furthermore, since all the probability density function involved are Gaussian, we use following formula for the entropy of Gaussian density function.

$$H(\rho(w)) = \frac{1}{2} \log((2\pi e)^N |\Sigma(t)|)$$

for $\rho(w)$ of the form

$$\rho(w) = \frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma(t)|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} w^\top \Sigma(t)^{-1} w\right)$$

Hence to compute the required transfer, we only need to know the covariance of $\rho(y'|y)$ and $\rho_{\mathcal{F}}(y'|y)$. The covariance matrix for the joint z' and z is given by

$$R_{z'z} = \begin{pmatrix} \Sigma_{z'} & \Sigma_{z'z} \\ \Sigma_{zz'} & \Sigma_z \end{pmatrix}, \quad \Sigma_z = \begin{pmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{xy}^\top & \Sigma_y \end{pmatrix}$$

where $\Sigma_{z'} = A\Sigma_z A^\top + \sigma^2 I$, $\Sigma_{zz'} = \Sigma_z A^\top$, $\Sigma_{z'z} = A\Sigma_z = \Sigma_{zz'}^\top$. Hence, we have

$$R_{y'y} = \begin{pmatrix} \Sigma_{y'} & \Sigma_{y'y} \\ \Sigma_{y'y}^\top & \Sigma_y \end{pmatrix}, \quad y'|y \sim \mathcal{N}(-, \Sigma_{y'} - \Sigma_{y'y} \Sigma_y^{-1} \Sigma_{y'y}^\top)$$

$$\begin{aligned} \Sigma_{y'} &= A_{yx} \Sigma_x A_{yx}^\top + A_y \Sigma_{xy} A_{yx}^\top + A_{yx} \Sigma_{xy} A_y^\top \\ &\quad + A_y \Sigma_y A_y^\top + \sigma^2 I \end{aligned}$$

$$\Sigma_{y'y} = A_{yx} \Sigma_{xy} + A_y \Sigma_y$$

Note that since the entropy is function of covariance, we ignore the explicit computation of mean for the underlying Gaussian random variable. After simplification, we obtain

$$\rho(y'|y) \sim \mathcal{N}(-, A_{yx}(\Sigma_x - \Sigma_{xy} \Sigma_y^{-1} \Sigma_{xy}^\top) A_{yx}^\top + \sigma^2 I).$$

For $\rho_{\mathcal{F}}(y'|y)$ we look at the system, $y' = A_y y + \sigma \xi_y$. Hence, we have $\rho_{\mathcal{F}}(y'|y) \sim \mathcal{N}(-, \sigma^2 I)$. Hence we have

$$T_{x \rightarrow y}(t) = \frac{1}{2} \log \frac{|A_{yx} \Sigma_y^s(t) A_{yx}^\top + \sigma^2 I|}{\sigma^2}$$

where $\Sigma_y^s(t) = \Sigma_x(t) - \Sigma_{xy}(t) \Sigma_y(t)^{-1} \Sigma_{xy}(t)^\top$.

To compute the information transfer from $x_1 \rightarrow y$, we need to compute covariance for $\rho_{\mathcal{F}}(y'|y)$. Hence, we look at the system where x_1 is absent. Hence, we look at

$$\begin{aligned} x_2' &= A_{x_2} x_2 + A_{x_2 y} y + \xi_{x_2} \\ y' &= A_{y x_2} x_2 + A_y y + \xi_y \end{aligned} \tag{17}$$

The expression for $\rho(y'|y)$ remains unchanged from the previous case. To compute $\rho_{\neq 1}(y'|y)$, we use system equation (17) and use procedure similar to one used in the derivation of $\rho_{\neq}(y'|y)$ with x replaced with x_1 . Hence, we have

$$\rho_{\neq 1}(y'|y) = \mathcal{N}(-, A_{yx_2}(\Sigma_y^s)_{yx_2}(t)A_{yx_2}^\top + \sigma^2 I) \quad (18)$$

where,

$$(\Sigma_y^s)_{yx_2}(t) = (\Sigma_{x_2}(t) - \Sigma_{x_2y}(t)\Sigma_y^{-1}(t)\Sigma_{x_2y}^\top(t)).$$

Hence,

$$[T_{x_1 \rightarrow y}]_t^{t+1} = \frac{1}{2} \log \frac{|A_{yx}\Sigma_y^s(t)A_{yx}^\top + \sigma^2 I|}{|A_{yx_2}(\Sigma_y^s)_{yx_2}(t)A_{yx_2}^\top + \sigma^2 I|} \quad (19)$$

□

The results from the above theorem can be used to provide general expression for information transfer between scalar state z_i and z_j for linear network system. In particular with no loss of generality we can assume $z_i = z_1$ and $z_j = z_2$, then the expression for $T_{z_1 \rightarrow z_2}$ can be obtained from (16) by defining

$$x_1 := z_1, \quad y = z_2, \quad x_2 := (z_3, \dots, z_N)$$

The formula for n -step transfer can also be derived in a similar manner, where one has to look at the n -step covariance matrix.

For linear systems with Gaussian noise, the one step zero transfer can be characterized by looking at system matrix A . In particular, we have the following theorem.

Theorem 8. $A_{z_j z_i} = 0$, iff $[T_{z_i \rightarrow z_j}]_t^{t+1} = 0$.

Proof. The proof follows from formula (16). □

3.3 Revisiting the examples

In this subsection we revisit the examples presented in section (2) and show that the new definition of information overcome some of the issues with directed information.

Example 1 revisited : For the three state example discussed in section 2, we compute the information transfer using our proposed definition of information transfer and the formula (16). In Fig. 4(a), we plot the information transfer over n time steps and we notice that the information transfer from $x_1 \rightarrow x_2$ and $x_1 \rightarrow x_3$ are identically zero.

Example 2 revisited : We compute the information transfer from $x_1 \rightarrow x_k$ and from $x_k \rightarrow x_N$ for $k = 15$. Since, x_1 is connected to x_k through series of k delays, as expected the information transfer $T_{x_1 \rightarrow x_k}$ is zero over $n < k$ time step and for $n = k$ the information transfer jumps from zero to non-zero value as shown in Fig. 4(b). Similarly information transfer from $x_k \rightarrow x_N$ also shows a sudden jump. This proves that our proposed definition of information transfer obeys the time constraints of transfer.

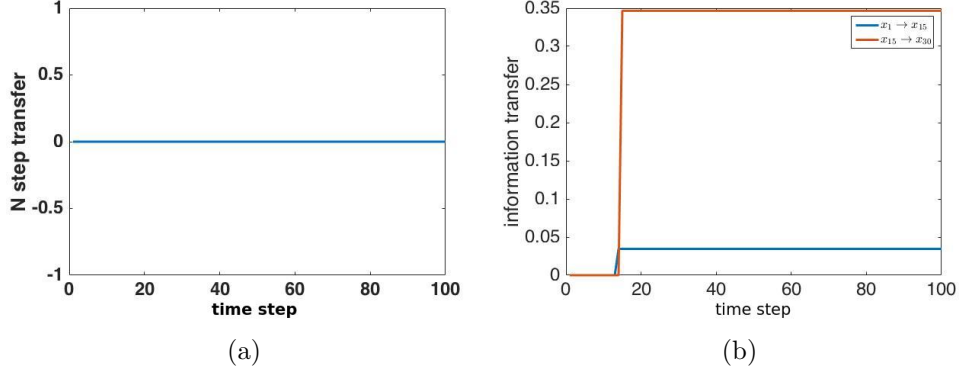


Figure 4: (a) N-step information transfer for example 1. (b) N-step information transfer for example 2.

4 Information Transfer in Linear Control Systems

In this section, we derive expression for information transfer in control dynamical system. In particular, we derive expressions for information transfer from state to output, input to state, and input to output. For the simplicity of discussion we also restrict our discussion to single input single output case.

We consider the following linear linear time invariant system with input and output.

$$\begin{aligned} z(t+1) &= Az(t) + Bu(t) \\ \theta(t) &= Cz(t) + \omega(t) \end{aligned} \quad (20)$$

where $z \in \mathbb{R}^N$ are the states, $A \in \mathbb{R}^{N \times N}$, B and C are of appropriate dimensions, $u(t) \in \mathbb{R}$ is the input, $\theta \in \mathbb{R}$ is output and $\omega(t)$ is the output noise, which is assumed to be zero mean i.i.d. Gaussian noise. For the simplicity of presentation, we will restrict our discussion to the case where the state space z is split in only two subspace i.e., $z = (x^\top, y^\top)^\top$ and the inputs and outputs are one dimensional. With this assumption, we have following splitting of the A , B , and C matrices.

$$A = \begin{pmatrix} A_x & A_{xy} \\ A_{yx} & A_y \end{pmatrix}, B = \begin{pmatrix} B_x \\ B_y \end{pmatrix}, C = (C_x \ C_y) \quad (21)$$

4.1 Information from Input to State

The evolution of the state x is $x' = A_x x + A_{xy} y + B_x u$. In deriving the formulas in this section, we think of the input $u(t)$ as a i.i.d. Gaussian variable such that

$$\mathbb{E}[u(t)u(t+k)^\top] = 0, \forall k \neq t \text{ and } \mathbb{E}[zu^\top] = 0.$$

Hence,

$$\Sigma_{x'|x} = A_{xy} \Sigma_x^s A_{xy}^\top + B_x \Sigma_u B_x^\top \quad (22)$$

where Σ_x^s is the Schur complement of Σ_x in Σ_z and Σ_u is the covariance of the input at time t .

When the input u is held frozen,

$$\Sigma_{x'|x}^u = A_{xy}\Sigma_x^s A_{xy}^\top \quad (23)$$

Hence the information transfer from u_1 to x is

$$T_{u \rightarrow x} = \frac{1}{2} \log \frac{|A_{xy}\Sigma_x^s A_{xy}^\top + B_x \Sigma_u B_x^\top|}{|A_{xy}\Sigma_x^s A_{xy}^\top|} \quad (24)$$

where $|\cdot|$ is the determinant.

4.2 Information from State to Output

In this subsection, we look at the information transfer from the states of the system to the output of the control system. For simplicity, we only look at the information transfer from the entire state space z to the output θ . The general case of the information transfer from any state z_i to any output θ_j for a MIMO system can be dealt with in similar manner.

From the output equation, we have

$$\Sigma_{\theta'|\theta} = C\Sigma_{z'}C^\top + \Sigma_\omega - (CA\Sigma_zC^\top)(C\Sigma_zC^\top)^{-1}(CA\Sigma_zC^\top)^\top$$

where $\Sigma_{z'} = A\Sigma_zA^\top + B\Sigma_uB^\top$ and Σ_ω is the covariance of the output noise.

When z is frozen, the output equation is

$$\theta_z(t) = \omega(t) \quad (25)$$

Hence, $\Sigma_{\theta'|\theta}^z = \Sigma_\omega$.

So the transfer is

$$T_{z \rightarrow \theta} = \frac{1}{2} \log \frac{|\Sigma_{\theta'|\theta}|}{|\Sigma_\omega|}$$

where

$$\Sigma_{\theta'|\theta} = C\Sigma_{z'}C^\top + \Sigma_\omega - (CA\Sigma_zC^\top)(C\Sigma_zC^\top)^{-1}(CA\Sigma_zC^\top)^\top$$

4.3 Information from Input to Output

As before, we address the case of SISO systems for simplicity. We have

$$\Sigma'_\theta = CA\Sigma_zA^\top C^\top + CB\Sigma_uB^\top C^\top$$

Hence,

$$\Sigma_{\theta'|\theta} = CA\Sigma_zA^\top C^\top + CB\Sigma_uB^\top C^\top + \Sigma_\omega - (CA\Sigma_zC^\top)(C\Sigma_zC^\top)^{-1}(CA\Sigma_zC^\top)^\top \quad (26)$$

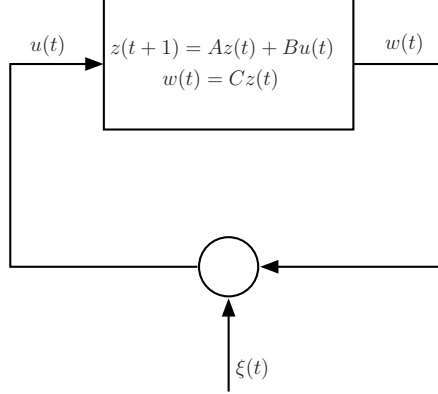


Figure 5: Feedback control system

When u is frozen, we have

$$\begin{aligned} z' &= Az \\ \theta &= Cz + \omega \end{aligned} \quad (27)$$

Hence,

$$\Sigma_{\theta'_{\mathcal{U}}|\theta_{\mathcal{U}}} = CA\Sigma_z A^\top C^\top + \Sigma_\omega - (CA\Sigma_z C^\top)(C\Sigma_z C^\top)^{-1}(CA\Sigma_z C^\top)^\top \quad (28)$$

Hence the transfer is

$$T_{u \rightarrow \theta} = \frac{1}{2} \log \frac{|\Sigma_{\theta'|\theta}|}{|\Sigma_{\theta'_{\mathcal{U}}|\theta_{\mathcal{U}}}|} \quad (29)$$

where $\Sigma_{\theta'|\theta}$ and $\Sigma_{\theta'_{\mathcal{U}}|\theta_{\mathcal{U}}}$ are given by equations (26) and (28) respectively.

4.4 Information Transfer in Feedback Control Systems

Information theory and feedback systems are interlinked and researchers have studied their interplay [19, 20, 21, 22]. Again, Massey [9] had shown that the feedback capacity (C_{fb}) satisfies

$$C_{fb} \leq \lim_{n \rightarrow \infty} \max_{\rho(X^n||Y^{n-1})} \frac{1}{n} I(X^n \rightarrow Y^n) \quad (30)$$

where $\rho(X^n||Y^{n-1})$ is the conditional distribution of X^n , causally conditioned on Y^{n-1} . Again, in [23], it has been proved that in a feedback system, the Bode sensitivity transfer function from the output to the input is equal to the average directed information from the output (w) to the input (u), where the average directed information from the output to the input is defined as

$$\lim_{n \rightarrow \infty} \frac{1}{n} I(w^n \rightarrow u^n) \quad (31)$$

In this section, we consider the following feedback control system (figure 5)

$$\begin{aligned} z(t+1) &= Az(t) + Bu(t) \\ w(t) &= Cz(t), \quad u(t) = w(t) + \xi(t) \end{aligned} \quad (32)$$

and we are interested in computing information flow from output to input of the feedback control system i.e., $T_{w \rightarrow u}$. This is the information transfer from the plant to itself. Towards this goal we write the feedback control system as follows:

$$\begin{aligned} z(t+1) &= (A + BC)z(t) + B\xi(t) \\ w(t) &= Cz(t), \quad u(t) = w(t) + \xi(t) \end{aligned} \quad (33)$$

Hence, $H_{\mathcal{W}}(\rho(u'|u)) = H(\rho(\xi'))$. With this, and theorem (5), next we show that the Bode sensitivity transfer function, \mathcal{S} , from w to u is same as the average information transfer from w to u .

Theorem 9.

$$\bar{T}_{x \rightarrow y} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log |\mathcal{S}(e^{j2\pi\theta})| d\theta = \sum_{i=1}^m \log |\lambda_i(A_u)|$$

where $\lambda_i(A_u)$ are the unstable eigenvalues of the open loop system.

Proof. From theorem (5), we have

$$\begin{aligned} \bar{T}_{x \rightarrow y} &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=1}^T (T_{x \rightarrow y})_0^i \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} [H(u^T) - H_{\mathcal{W}}(u^T)] = \lim_{T \rightarrow \infty} \frac{1}{T} [H(u^T) - H(\xi^T)] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \log \frac{|\Sigma_{u^T}|}{|\Sigma_{\xi^T}|} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log \frac{|u(e^{j2\pi\theta})|^2}{|\xi(e^{j2\pi\theta})|^2} d\theta \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \log |\mathcal{S}(e^{j2\pi\theta})| d\theta = \sum_{i=1}^m \log |\lambda_i(A_u)| \end{aligned}$$

where $\lambda_i(A_u)$ are the unstable poles of A .

□

Hence, the average directed information and average information transfer are the same for the above feedback control system. This theorem, along with the fact that the channel capacity of a feedback channel is related to directed information, presents further evidence that our formulation of information transfer, though developed from dynamical systems point of view, is consistent with already existing concepts in information theory and control theory. We feel this will allow us to study the concepts of information theory, control theory and dynamical systems from a more fundamental and common point of view.

Example 10. Consider the feedback system given in figure 5 with

$$A = \begin{pmatrix} 4 & 2 \\ 0 & 0.7 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = [-3.5 \quad 0]$$

We also assume that the noise ω is i.i.d. zero mean unit variance Gaussian white noise. The open loop system has unstable pole at 4. Hence the Bode integral of the sensitivity

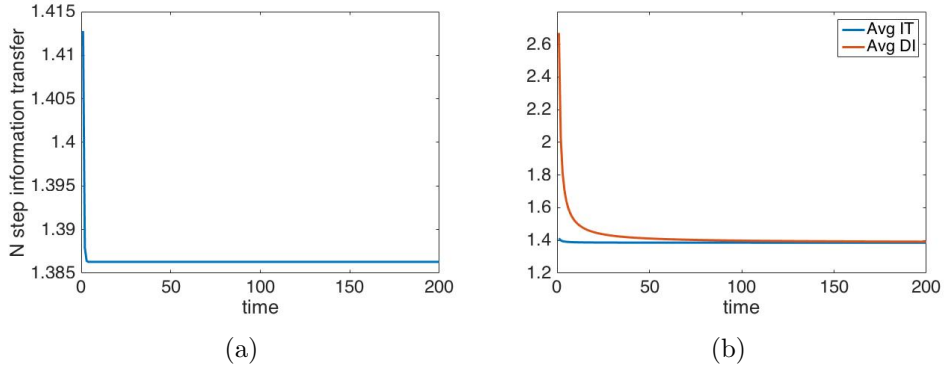


Figure 6: (a) N-step information transfer; (b) Average information transfer and average directed information

transfer function is $\log(4) = 1.3863$. Figure 6(a) shows the N-step information transfer from the output of the system to the input of the system. The steady state value of the information transfer is 1.3863. Similarly, the average information transfer from the output to the input is also 1.3863. This is shown in figure 6(b). In this figure we also plot the average directed information transfer from the output to input. Here again we observe that the average information transfer converges to the Bode integral of the sensitivity transfer function much faster than the average directed information.

5 Conclusion

In this paper, we discussed how directed information fails to capture the causal structure in a dynamical system and provided a new measure of causality based on a new definition of information transfer and have shown that it captures the intuitions of causality. The new definition of information, which is based on dynamical system setting, has been generalized to define information transfer between the various signals in a control dynamical system. This generalization resulted in connecting the information transfer with Bode integral of the sensitivity transfer function from the output to the input in a feedback control system.

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